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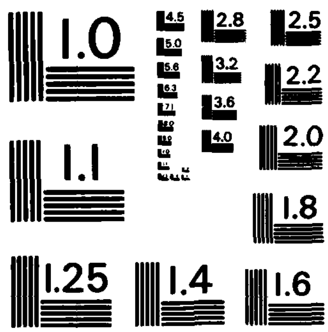

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DYNAMICS IN PARABOLIC EQUATIONS - AN EXAMPLE

Jack K. Hale

Division of Applied Mathematics
Brown University, Providence, R. I. 02912 U.S.A.

Abstract: For a parabolic equation, results are given about the orbits which connect equilibrium points. The approach is based on the theory of dynamical systems and the maximum principle.

1. The basic problem.

A system of nonlinear parabolic equations defined on a bounded domain generate a nonlinear semigroup of transformations $\{T_\mu(t), t\geq 0\}$ on some Banach space X. The parameter μ is supposed to represent the ellipti_ operators in the parabolic equation, the region Ω where the equation is defined, the nonlinear functions in the equation and the boundary conditions. The basic problem is to study how the qualitative properties of the orbits defined by $T_\mu(t)$ depend on the parameter μ.

Because $T_\mu(t)$ arises from parabolic equations on a bounded domain, one generally has

$$T_\mu(t),\ t > 0, \text{ is a completely continuous operator} \tag{1.1}$$

$$T_\mu(t) \text{ is one-to-one for all } t. \tag{1.2}$$

In particular, (1.1) implies that every bounded orbit belongs to a compact set. Thus, the ω-limit set of a bounded orbit is a nonempty, compact, connected set of X which is invariant under the semigroup $T_\mu(t)$. We say a set M in X is <u>invariant</u> for $T_\mu(t)$ if, for any $\varphi \in M$, there is a function $x(t,\varphi)$ defined for $t \in \mathbb{R}$, $x(0,\varphi) = 0$, $x(t,\varphi) \in M$, $t \in \mathbb{R}$, and, for any $\tau \in \mathbb{R}$, $T_\mu(t)x(\tau,\varphi) = x(t+\tau,\varphi)$, $t \geq 0$. We call $x(t,\varphi)$ a <u>backward extension</u> of φ and write $x(t,\varphi) = T_\mu(t)\varphi$ for $t \in \mathbb{R}$. Under

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condition (1.2), backward extensions are unique. If a backward extension of φ exists and belongs to a compact set, then the α-limit set exists and is nonempty, compact, connected and invariant (see [Ha2]).

An important role in the theory is played by the set,

$$A_\mu = \{\varphi \in X: T_\mu(t)\varphi \text{ is defined and bounded for } t \in \mathbb{R}\}, \tag{1.3}$$

which contains the limit points of all bounded orbits as well as much more. In fact, one has the following elementary result.

Lemma 1.1. <u>If A_μ is compact, then A_μ is the maximal compact invariant set of X. Furthermore, (1.2) implies that $T_\mu(t)$ is a group on A_μ.</u>

One way to ensure that A_μ is compact is to assume the existence of invariant regions for the parabolic system (see [S1]). Another weaker condition is to assume that the system is <u>point dissipative</u>; that is, there is a bounded set $B \subset X$ such that, for any $\varphi \in X$, there is a $t_0 = t_0(\varphi, B)$ such that $T_\mu(t)\varphi \in B$ for $t \geq t_0$. This condition implies the following result.

Theorem 1.2. <u>If $T_\mu(t)$, $t > 0$, is completely continuous and point dissipative, then there is a maximal compact invariant set A_μ for $\{T_\mu(t), t \geq 0\}$, A_μ is uniformly asymptotically stable and attracts bounded sets of X; that is, for any bounded set U in X, $\operatorname{dist}(T_\mu(t)U, A_\mu) \to 0$ as $t \to \infty$. If $T_\mu(t)$ is one-to-one, then $T_\mu(t)$ is a continuous group on A_μ.</u>

The basic steps in the proof are as follows. One first proves that there is a compact set K in X such that for any compact set H in X, there is a neighborhood V of H such that $\operatorname{dist}(T_\mu(t)V, K) \to 0$ as $t \to \infty$. This property implies there is a maximal compact invariant set A_μ which is uniformly asymptotically stable and attracts neighborhoods of compact sets. The compactness of $T_\mu(t)$ for $t > 0$ can be used to complete the proof.

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The above theorem is actually true for much more general semigroups $\{T_\mu(t), t \geq 0\}$. In fact, one need only require that $T_\mu(t)$ is an α-contraction for $t > 0$. This permits applications to a much broader class of problems including some hyperbolic systems (see [Ha3]). We do not consider such general cases here and refer the reader to [Ha1] for a complete proof of the above result as well as historical references. The discussion is also contained in [Ha2]. However, an error in the statement of one result in [Ha2] requires a reordering of the material and this reordering can be found in [Ha1].

We remark that it is not necessary to consider all of X to obtain an interesting invariant set A_μ. One could have a subset Y of X which is positively invariant with respect to $T_\mu(t)$. Point dissipative is then defined relative to the intersection of bounded sets with Y.

The basic problem in the qualitative theory of parabolic equations is to study how A_μ and the flow on the invariant set A_μ change with the parameter μ.

Before giving specific examples which illustrate the above remarks, let us state another interesting implication of point dissipativeness, a proof of which can be found in [HL 1].

Theorem 1.3. If $T_\mu(t)$ is completely continuous for $t > 0$ and point dissipative, then there is an equilibrium point of $T_\mu(t)$; that is, there exists a φ in X such that $T_\mu(t)\varphi = \varphi$ for all t.

When $T_\mu(t)$ is the semigroup generated by a system of parabolic equations, Theorem 1.3 implies the existence of a solution of the corresponding elliptic boundary value problem. For some types of problems, this approach could be easier to obtain the existence of solutions of the elliptic system (see [A1],[A2],[A3],[K1]).

2. An example. In this section, we consider the simplest nontrivial parabolic equation. In spite of its simplicity, there are several unanswered questions whose solution would lead to a better understanding of the role of diffusion in the dynamics.

Consider the scalar equation

$$\begin{aligned} &u_t = u_{xx} + \lambda f(u), \quad 0 < x < \pi, \\ &u = 0 \quad \text{at} \quad x = 0,\pi \end{aligned} \tag{2.1}$$

with $\lambda > 0$ being a real parameter and $f(u)$ being a given nonlinear function of u. If

$$V(\varphi) = \int_0^\pi [\varphi_x^2 - \lambda F(\varphi)]dx, \quad F(u) = \int_0^u f, \tag{2.2}$$

and $u(t,x)$ is a solution of (2.1), then

$$\frac{d}{dt}V(u(t,x)) = -\int_0^\pi u_t^2 dx \le 0 \tag{2.3}$$

Theorem 2.1. If

$$F(u) \to -\infty \quad \text{as} \quad u \to \pm\infty \tag{2.4}$$

then Eq. (2.1) generates a C_0-semigroup $T_\lambda(t)$, $t \geq 0$, on $X = H_0^1(0,\pi)$, each orbit is bounded and has ω-limit set as an equilibrium point. There is a maximal compact invariant set A_λ for $T_\lambda(t)$ which has the stability properties mentioned in Theorem 1.2. Finally, if $\varphi \in A_\lambda$, then the α-limit set of φ is an equilibrium point.

The equilibrium points of (2.1) are the solutions of the equation

$$\begin{aligned} u_{xx} + \lambda f(u) = 0 \quad & 0 < x < \pi \\ u = 0 \quad \text{at} \quad & x = 0,\pi \end{aligned} \tag{2.5}$$

That (2.1) generates a C_0-semigroup is given in Henry [He 1]. The fact that the ω- and α-limit sets must be a single equilibrium point has been proved by a number of people (see Zelenyak [Z 1], Matano [M 2], Hale and Massat [HM 1]). Relation (2.4) implies the set of equilibrium points is bounded. Since every orbit approaches an equilibrium point, one obtains point dissipative.

An equilibrium point u_0 is hyperbolic if no eigenvalue of the operator $\partial^2/\partial x^2 + \lambda f'(u_0)$ on X is zero and it is called stable (hyperbolic) if all eigenvalues are negative. The unstable manifold $W^u(u_0)$ is the set of $\varphi \in X$ such that $T_\mu(t)\varphi$ is defined for $t \leq 0$ and $\to u_0$ as $t \to -\infty$. The stable manifold $W^s(u_0)$ is the set of $\varphi \in X$ such that $T_\mu(t)\varphi \to u_0$ as $t \to \infty$. The set $W^u(u_0)$ is an embedded submanifold of X of finite dimension m (m being the number of positive eigenvalues of the above operator). The set $W^s(u_0)$ is an embedded submanifold of codimension m (see [He 1]). These manifolds are tangent at u_0 to the stable and unstable manifolds of the linear operator $\partial^2/\partial x^2 + f'(u_0)$ on X.

The following remark is a simple but important consequence of Theorem 2.1.

Corollary 2.2. If (2.4) is satisfied and there are only a finite number of hyperbolic equilibrium points $\varphi_1, \varphi_2, \ldots, \varphi_k$ of (1.1) with each being hyperbolic, then

$$A_\lambda = \bigcup_{j=1}^{k} W^u(\varphi_j).$$

Corollary 2.2 states that A_λ is the union of a finite number of finite dimensional manifolds. The complete flow on A_λ seems to be difficult to describe in the general case. However, some nontrivial information is easily obtained if we make additional hypotheses.

Theorem 2.3. If the conditions of Corollary 2.2 are satisfied and $\varphi_1(x) < \varphi_2(x) < \cdots < \varphi_k(x)$, $0 < x < \pi$, and $\dim W^u(\varphi_j) \le 1$, then k is odd, $k = 2p+1$, φ_{2j+1} is stable hyperbolic and φ_{2j} is unstable for all j with the ω-limit set (α-limit set) of φ_{2j} being $\varphi_{2j+1}(\varphi_{2j-1})$.

Proof: If φ_i is unstable, then $\dim W^u(\varphi_i) = 1$. Since $W^u(\varphi_i)$ is tangent at φ_i to the line spanned by the eigenfunction corresponding to the positive eigenvalue of $\partial^2/\partial x^2 + f'(\varphi_i)$ on X and this eigenfunction has no zeros on $(0,\pi)$, it follows that there is a solution $u(t,x)$ of (2.1) defined for $t \in \mathbb{R}$ with $\lim_{t\to-\infty} u(t,x) = \varphi_i(x)$ and $u(t,x) > \varphi_i(x)$ for t sufficiently negative. The maximum principle and our hypothesis on φ_{i+1} implies that $\varphi_i(x) < u(t,x) < \varphi_{i+1}(x)$ for $0 < x < \pi$. Thus, $\lim_{t\to\infty} u(t,x) = \varphi_{i+1}(x)$. The same argument shows there is a $\psi \in W^u(\varphi_i)$ such that the ω-limit set of ψ is φ_{i-1}. Since A_λ is stable, φ_1 and φ_k are stable hyperbolic and the theorem is proved.

Corollary 2.4. Under the hypotheses of Theorem 2.3, the global dynamics of (2.1) is determined by the local bifurcation of equilibrium points.

Theorem 2.3 has immediate application to positive solutions of Eq. (2.1). In fact, if $X^+ = \{\varphi \in X: \varphi(x) \ge 0,\ 0 \le x \le \pi\}$, and $T_\lambda(t): X^+ \to X^+$, then we get the maximal compact invariant set A_λ^+ in X^+. In this case, the dim $W^u(\varphi) \le 1$ for any equilibrium point. Thus, if the equilibrium points are hyperbolic, then the conclusion of Theorem 2.3 is true. For another discussion of this latter example using the Conley index, see Smoller [S 1].

Remark 2.5. Using the same proof as above, one observes that the conclusions of Theorem 2.3 are true if u_{xx} is replaced by Δu and $(0,\pi)$ is replaced by a bounded open set $\Omega \subset \mathbb{R}^N$.

We also remark that the strong conclusion in Theorem 2.3 and Corollary 2.4 are not consequences of only $\dim W^u(\varphi_j) \le 1$. The parabolicity is used in an essential way. The reader is referred to Hale and Rybakowski [HR 1] for an example of a scalar gradient-like delay equation where the conclusions of Theorem 1.4 are not true even though the dimension of the unstable manifold of each equilibrium point is ≤ 1.

Let us now return to the general discussion. When the equilibrium points of (2.1 are hyperbolic, Corollary 2.2 gives A_λ as the union of the unstable manifolds of the equilibrium points. The complete dynamics on A_λ will only be known when we know the specific way in which the equilibrium points are

connected to each other by orbits. In an effort to learn more about this, we introduce some other invariant sets of A_λ.

Let

$$A_\lambda^j = \{\varphi \in A_\lambda : \{T_\lambda(t)\varphi, t \in \mathbb{R} \text{ as well as its } \alpha\text{- and } \omega\text{-limit sets have exactly } j \text{ zeros in } (0,\pi)\}. \tag{2.6}$$

Each set A_λ^j is compact and invariant.

<u>Lemma 2.6.</u> <u>Suppose</u> $f(0) = 0$. <u>If</u> $\varphi \in A_\lambda \setminus \bigcup A_\lambda^p$ <u>with</u> α-<u>limit set in</u> A_λ^j <u>and</u> ω-<u>limit set in</u> A_λ^k, <u>then</u> $j > k$.

<u>Proof.</u> Suppose such a φ exists. Since, as $t \to -\infty$, $T_\lambda(t)\varphi \to \psi \in A_\lambda^j$, an equilibrium point with j zeros in $(0,\pi)$ and since the zeros of ψ are simple, it follows that $T_\lambda(t)\varphi$ has j zeros in $(0,\pi)$ for $t \leq -\tau$ with τ sufficiently large. Also, $T_\lambda(t)\varphi$ has $j + 1$ extreme values in $(0,\pi)$. Also, $T_\lambda(t)\varphi \to \eta \in A_\lambda^k$, with k simple zeros and $k + 1$ extreme values in $(0,\pi)$. Thus, if $\bar{\varphi} = T_\lambda(-\tau)\varphi$, then $T_\lambda(t)\bar{\varphi} \to \eta$ and the results of Matano [M 1] imply that the number of extreme values of $\bar{\varphi}$ must be $\geq$ the number of extreme values of η. This implies $j \geq k$. But since $j \neq k$, we have the result.

The condition $f(0) = 0$ is, in general, necessary to obtain the conclusion in Lemma 2.6. If $f(0) \neq 0$, then the best general result would be $j \geq k - 2$ (see [M 1]).

Let $M_i \in A_\lambda$, $i = 1,2,\ldots,p$, be compact invariant sets in A_λ. Following Conley [C 1], we say $\{M_i\}$ is a <u>Morse decomposition</u> of A_λ if $\varphi \in A_\lambda \setminus \bigcup_i M_i$ implies there are integers $j > k$ such that the ω-limit set of φ is in M_k and the α-limit set of φ is in M_j.

Under the assumptions of Lemma 2.6, $f(0) = 0$ implies $u = 0$ is an equilibrium solution of (1.1). Thus, the set $\{0\}$ must be included with the A_λ^j in (2.6) in order to have any hope for a Morse decomposition. Since Lemma 2.6 says nothing about $\{0\}$, more detailed information is needed about the behavior of the solutions near $u = 0$. However, we can state the following general consequence of Lemma 2.6.

<u>Theorem 2.7.</u> <u>Suppose</u> $f(0) = 0$ <u>and the sets</u> A_λ^k, $k=0,1,\ldots,p$, <u>are defined in (1.6) and let</u> $A_\lambda^{p+1} = \{0\}$. <u>If no orbit in</u> A_λ <u>has</u> ω-<u>limit set</u> A_λ^{p+1}, <u>then the sets</u> $\{A_\lambda^k, k=0,1,\ldots,p+1\}$ <u>form a Morse decomposition for</u> A_λ.

Other variants of Theorem 2.7 can easily be given if one knows more details about the stable and unstable manifold of $u = 0$ in A_λ.

To introduce some other ideas, let us consider the special case of equation (2.1) studied extensively by Chafee and Infante [CI 1] and Henry [He 1], where

$$\begin{aligned} &f(0) = 0, \ f'(0) = 1 \\ &\limsup f(u)/u \le 0, \quad uf''(u) > 0 \ \text{ if } \ u \ne 0. \end{aligned} \tag{2.7}$$

Theorem 2.8. If f satisfies (2.7) and $\lambda \in (n^2,(n+1)^2)$, n an integer, then there are exactly $2n+1$ equilibrium points $\alpha_\infty = 0$, α_j^+, α_j^-, $j=0,1\ldots,n-1$, where α_j^+, α_j^- have j zeros in $(0,\pi)$, $\dim W^u(\alpha_j^\pm) = j$, $0 \le j \le n-1$, $\dim W^u(\alpha_\infty) = n$ and

$$A_\lambda = (\textstyle\bigcup_j W^u(\alpha_j^\pm)) \cup W^u(\alpha_\infty).$$

With the A_λ^k defined as in (2.6) and $A_\lambda^\infty = \{\alpha_\infty = 0\}$, the set $\{A_\lambda^\infty, A_\lambda^k, \ k=0,1,2,\ldots,n-1\}$ is a Morse decomposition of A_λ.

Proof. The first part may be found in [He 1]. We prove that $\alpha_\infty = 0$ is completely unstable in A_λ. In fact, any function in $W^s(\alpha_\infty)$ except zero must have at least n zeros and $n+1$ extrema. The results of Matano [M 1] imply the assertion. Theorem 2.7 completes the proof.

Remark 2.9. The function $\alpha_j^+[\alpha_j^-]$ is uniquely specified by requiring that $d\alpha_j^+(0)/dx > 0$ $[d\alpha_j^-(0)/dx < 0]$. Also, the properties of the orbits in the phase plane imply $\alpha_{2j+1}^+(\pi-x) = \alpha_{2j+1}^-(x)$.

Although Theorem 2.8 gives some information about orbits which connect equilibrium points, it is very imprecise. One would like to have a better understanding of exactly which connections exist. Henry [He 1] has given a complete answer to this problem for $n = 0,1,2,3$, for $f(u)\ldots f(-u)$. We will reprove this result in a slightly different way and, at the same time, give some more information which will hopefully point out the difficulties involved for larger n.

For the statement of the next result, some additional notation is needed. The set $W^s(\alpha_\infty)$ is an embedded submanifold of X (see [He 1,p.155 ff]) of codimension n and $W^s(\alpha_j^\pm)$ is an embedded submanifold of codimension j. The set $W^u(\alpha_\infty)$ is

tangent at α_∞ to the linear manifold of X spanned by eigenfunctions of the operator $\partial^2/\partial x^2 + f'(\alpha_\infty)$ on X corresponding to the positive eigenvalues $0 < \lambda_{n-1,\infty} < \lambda_{n-2,\infty} < \ldots < \lambda_{0,\infty}$. The

see Correction → eigenfunction $\lambda_{j,\infty}$ also has j zeros in $(0,\pi)$ for each j.

→ One can construct an imbedded $(n-j)$-dimensional submanifold $W^u_j(\alpha_\infty) \subset W^u(\alpha_\infty)$ for $j=0,1,\ldots,n-1$ which consists of all orbits which approach α_∞ as $t \to -\infty$ with an exponential rate $\leq \lambda_{j,\infty}$. Let

$$W^{u*}_j(\alpha_\infty) = W^u(\alpha_\infty) \setminus W^u_j(\alpha_\infty). \tag{2.8}$$

In the same way, for any $j \leq k$, $k = 0,1,\ldots,n-1$, one can define

$$W^{u*}_j(\alpha^\pm_k) = W^u(\alpha^\pm_k) \setminus W^u_j(\alpha^\pm_k). \tag{2.9}$$

The set $W^{u*}_j(\alpha^\pm_k)$ is a $(k-j)$ dimensional submanifold of $W^u(\alpha^\pm_k)$.

We also let

$$C^{n,k}_\lambda = \{\varphi \in A_\lambda : \lim_{t\to-\infty} T_\lambda(t)\varphi \in A^n_\lambda,\ \lim_{t\to\infty} T(t)\varphi \in A^k_\lambda\} \tag{2.10}$$

and let $\gamma(\varphi,\psi)$ designate an orbit whose α-limit set is φ and ω-limit set is ψ.

Lemma 2.10. If f satisfies (2.7), is odd and $\lambda \in (n^2,(n+1)^2)$, then

$$\begin{aligned} C^{k,k}_\lambda &= \emptyset \\ C^{\infty,0}_\lambda &= W^{u*}_1(\alpha_\infty) \supseteq \{\gamma(\alpha_\infty,\alpha^\pm_0)\} \\ C^{k,0}_\lambda &= W^{u*}_1(\alpha^+_k) \cup W^{u*}_1(\alpha^-_k) \supseteq \{\gamma(\alpha^\pm_k,\alpha^\pm_0)\}. \end{aligned}$$

Proof. From the oddness of f and the existence of the Liapunov functional V in (2.2), it is easy to see that $C^{k,k}_\lambda = \emptyset$.

We now show that there is a special type of orbit from α_∞ to α^+_0. As remarked before, the set $W^u(\alpha_\infty)$ is tangent at α_∞ to the linear manifold of X spanned by the eigenfunctions of the operator $\partial^2/\partial x^2 + f'(\alpha_\infty)$ on X corresponding to the positive eigenvalues $0 < \lambda_{n-1,\infty} < \ldots < \lambda_{0,\infty}$. These eigenfunctions have, respectively, $n-1,\ldots,1,0$ zeros in $(0,\pi)$. Thus, there is a $\varphi \in X$ such that $T_\lambda(t)\varphi \to \alpha_\infty$ as $t \to -\infty$ and $T_\lambda(t)\varphi(x) > 0$ on $(0,\pi)$ for $t \leq -\tau$ for τ sufficiently large. The maximum principle implies $T_\lambda(t)\varphi(x) > 0$ for all t and, thus, the ω-limit set is an equilibrium point with no zeros in $(0,\pi)$; that is, α^+_0. The same type of argument shows that $\gamma(\alpha_\infty,\alpha^-_0)$ exists.

see Correction → Continuity with respect to initial data and the fact that

α_0^+, α_0^- are stable implies the assertion that $C_\lambda^{\infty,0} \supseteq W_1^{u*}(\alpha_\infty)$.

To show $\gamma(\alpha_j^+, \alpha_0^+)$, $j \geq 1$, exists, one proceeds in the same way as above to find a $\varphi \in X$ such that $T_\lambda(t)\varphi \to \alpha_j^+$ as $t \to -\infty$ and $T_\lambda(t)\varphi(x) > \alpha_j^+(x)$ on $(0,\pi)$. Then the ω-limit set of φ must be $> \alpha_j^+(x)$ on $(0,\pi)$. From the manner in which the $\alpha_j^\pm(x)$ are constructed from phase plane analysis, it follows that no equilibrium point is larger than $\alpha_j^+(x)$ on $(0,\pi)$ except α_0^+. In the same way, one obtains $\gamma(\alpha_j^+, \alpha_0^-), \gamma(\alpha_j^-, \alpha_0^+), \gamma(\alpha_j^-, \alpha_0^-)$. A

ee Correction use of continuity in the same way as above shows that $C_\lambda^{k,0} \supseteq W_1^{u*}(\alpha_k^+) \cup W_1^{u*}(\alpha_k^-)$. It remains to show equality.

Since α_0^+, α_0^- are stable, it follows that the set

$$X_0 = X \setminus (W^s(\alpha_0^+) \cup W^s(\alpha_0^-)) \tag{2.11}$$

is invariant under $T_\lambda(t)$, has no interior and X_0 contains all the equilibrium points $\alpha_j^\pm$, $j \geq 1$, and α_∞ as well as their stable manifolds. Furthermore, if δ is any continuous curve joining α_0^+, to α_0^-, then $\delta \cap X_0 \neq \emptyset$.

Now suppose $C_\lambda^{\infty,0}$ contains an orbit $\tilde{\gamma}$ from α_∞ to α_0^+ which is not the ones constructed above. Then the oddness of f implies $-\tilde{\gamma}$ goes from α_∞ to α_0^-. The orbits $\gamma, -\gamma \in W_1^u(\alpha_\infty)$ where $W_1^u(\alpha_\infty)$ is defined in (2.8). The openness of $W^s(\alpha_0^+), W^s(\alpha_0^-)$ implies that, in a neighborhood U of α_∞,

ee Correction the set $X_0 \cap U$ is a submanifold of codimension 1. This contradicts the property that $\delta \cap X_0 \neq \emptyset$ for any continuous curve δ joining α_0^+ to α_0^-. Doing the same argument for $\alpha_k^\pm$ completes the proof of the theorem.

Lemma 2.11. Suppose f satisfies (2.7), $\lambda \in (n^2, (n+1)^2)$ and ψ is an equilibrium point with $j+k$ zeros and φ is an equilibrium point with j zeros in $(0,\pi)$. Then any orbit $\gamma(\psi,\varphi)$ going from ψ to φ must belong to $W_j^u(\psi)$.

Proof. Let φ be an equilibrium point with $j+k$ zeros and let ψ be an equilibrium point with j zeros. Then the manner in which the equilibrium points are constructed from the phase plane implies that $\psi - \varphi$ has j zeros in $(0,\pi)$ and $j+1$ extreme values. If $u = \varphi + v$ in (2.1), then $W^u(0)$ for the new equation has dimension $j+k$ with the basis for the tangent manifold at zero being given by functions with $0,1,\ldots,j+k-1$ zeros in $(0,\pi)$. Since $\psi - \varphi$ is an equilibrium point for the new equation with $j+1$ extreme values in $(0,\pi)$, one can apply

see Correction → the results of Matano [M 1] to obtain the first assertion in the Lemma. The last assertion is obtained in the same way.

To point out implications of this lemma, define

$$S_\lambda^0 = A_\lambda,$$
$$S_\lambda^k = S_\lambda^{k-1} \setminus [(W^s(\alpha_{k-1}^+) \cup W^s(\alpha_{k-1}^-)) \cap A_\lambda], \quad k=1,2,\ldots,n-1. \tag{2.12}$$

The points α_0^+, α_0^- are stable in S_λ^0. Also, Lemma 2.11 (or Lemma 2.10 with $k=1$) imply that α_1^+, α_1^- are stable in S_λ^1. Furthermore, Lemma 2.10 implies these are the only stable points in S^1 and $\dim[W^u(\alpha_k^\pm) \cap S_\lambda^1] = k-1$, $k=2,3,\ldots,n-1$, $\dim[W^u(\alpha_\infty) \cap S_\lambda^1 = n-1$. Lemma 2.11 and the symmetry hypothesis implies $C_\lambda^{2,1} = W_1^u(\alpha_2^+) \cup W_1^u(\alpha_2^-) \supseteq \{\gamma(\alpha_2^+,\alpha_1^+), \gamma(\alpha_2^+,\alpha_1^-), \gamma(\alpha_2^-,\alpha_1^+), \gamma(\alpha_2^-,\alpha_1^-)\}$.

In general, we can assert that α_j^+, α_j^- are stable in S_λ^j for every j. This is a consequence of Lemma 2.11. For $j=n-1$, this implies $\alpha_{n-1}^+, \alpha_{n-1}^-$ is stable in S_λ^{n-1}. The only other

see Correction → equilibrium point in S^{n-1} is α_∞ and Lemma 2.11 implies
→ $C_\lambda^{\infty,n-1} = W_{n-1}^u(\alpha_\infty) = \{\gamma(\alpha_\infty,\alpha_{n-1}^+), \gamma(\alpha_\infty,\alpha_{n-1}^-)\}$.

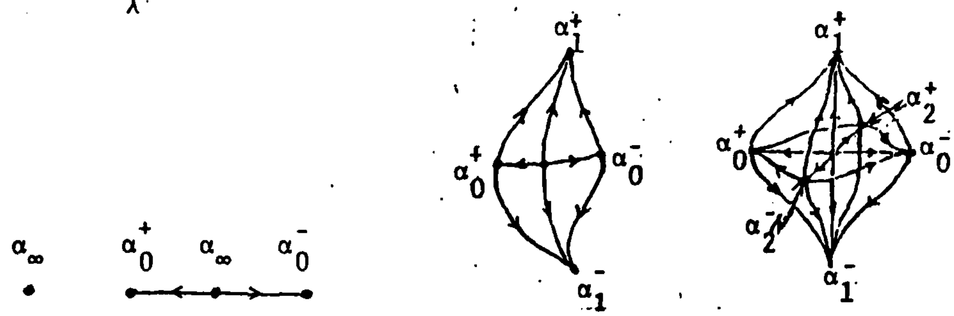
For $j=2,3,\ldots,n-2$, we are unable to analyze the fine structure of the flow at this time. To see the difficulties, suppose $j = 2$. Then α_2^+, α_2^- are stable in S_λ^2. From what has been proved up to this point, there is the possibility that α_3^+, α_3^- are also stable in S_λ^2; that is, the two dimension unstable manifold of these points in S_λ^1 have ω-limit set $\{\alpha_1^+, \alpha_1^-\}$. If such things can occur, then the flow can be very complicated and can change its qualitative properties without going through a local bifurcation.

Summarizing the nicest part of what has been proved, we obtain the following result of Henry [He 1].

Theorem 2.12. If f satisfies (2.7), is odd and $\lambda \in (n^2,(n+1)^2)$, $n=0,1,2,3$, then

$$C_\lambda^{m,k} = W_k^{u*}(\alpha_m^+) \cup W_k^{u*}(\alpha_m^-) \supseteq \{\gamma(\alpha_m^\pm,\alpha_k^\pm)\}, \quad n-1 \geq m > k \geq 0$$
$$C_\lambda^{\infty,k} = W_k^{u*}(\alpha_\infty) \geq \{\gamma(\alpha_\infty,\alpha_k^\pm)\}, \quad n-1 \geq k \geq 0$$
$$A_\lambda^k = \{\alpha_k^+,\alpha_k^-\}$$
$$A_\lambda = \bigcup_{m>k} (A_\lambda^k \cup C_\lambda^{m,k}) \cup (A_\lambda^\infty \cup C_\lambda^{\infty,k})$$

Theorem 2.12 says that A is like a ball of n-dimensions if $\lambda \in (n^2,(n+1)^2)$, $n=0,1,2,3$. The pictures of A_λ and the flow on A_λ are shown below.



n = 0 n = 1 n = 2 n = 3

For $n \leq 3$, Theorem 2.12 asserts that the changes that occur in the dynamics on A_λ for $\lambda < 16$ come about only through local bifurcation. A proof that the same situation prevails for all λ or a counterexample would be equally interesting.

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e insert [Ha 1]

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Added in proof: The details of the proof of that part of the proof of Lemma 2.10 which states that $C_\lambda^{\infty,0} \supseteq W_1^u(\alpha_\infty)$ are not as easily supplied as indicated. The same applies to $C_\lambda^{k,0}$. Thus, this must remain as a conjecture.

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August 27, 1982

Corrections and Insert

for

DYNAMICS IN PARABOLIC EQUATIONS - AN EXAMPLE

by Jack K. Hale

p. 8, lines 6 → 7

eigenfunction for $\lambda_{j,\infty}$ has j zeros in $(0,\pi)$ for each j. One can construct an imbedded $(n-j)$-dimensional submanifold

correction p. 2 starting on line → 42

The above theorem is actually true for much more general semigroups $\{T_\mu(t), t \geq 0\}$. One needs only that $T_\mu(t)$ is compact dissipative and an α-contraction for $t > 0$. This permits applications to a much broader class of problems including some hyperbolic systems (see [HA 3]). We do not consider such general cases here and refer the reader to [Ha 1] for a complete proof of the above result as well as historical references. The discussion is also contained in [Ha 2]. However, an error in the statement of one result in [Ha 2] requires a reordering of the material and this reordering can be found in [Ha 1].

Corrections and Insert

p. 10 lines → 3 the results of Matano [M 1] to obtain the assertion in the
→ 4 Lemma.

p. 9 line → 15 use of an argument similar to the above shows that $C_\lambda^{k,0} \supseteq$

p. 10. lines → 24 equilibrium point in S^{n-1} is α_∞ and, thus,
→ 25 $C_\lambda^{\infty,n-1} = W_{n-2}^{u*}(\alpha_\infty) = \{\gamma(\alpha_\infty,\alpha_{n-1}^+),\gamma(\alpha_\infty,\alpha_{n-1}^-)\}$.

p. 9 lines → 34 the set $X_0 \cap U$ is a proper subset of a submanifold of codimen-
→ 35 sion 1. This contradicts $\delta \cap X_0 \neq \emptyset$ for any continuous curve

p. 11 insert →

[Ha 1] J.K. Hale, Some recent results on dissipative processes. p.152-172 in Lecture Notes in Math. Vol. 799 (Ed.A.F. Izé), Springer-Verlag, 1980.

p. 8 line → 52 Continuity, the Lyapunov functional, X_0 having no interior and